Lecture 11

Transmission Lines

11.1 Transmission Line Theory

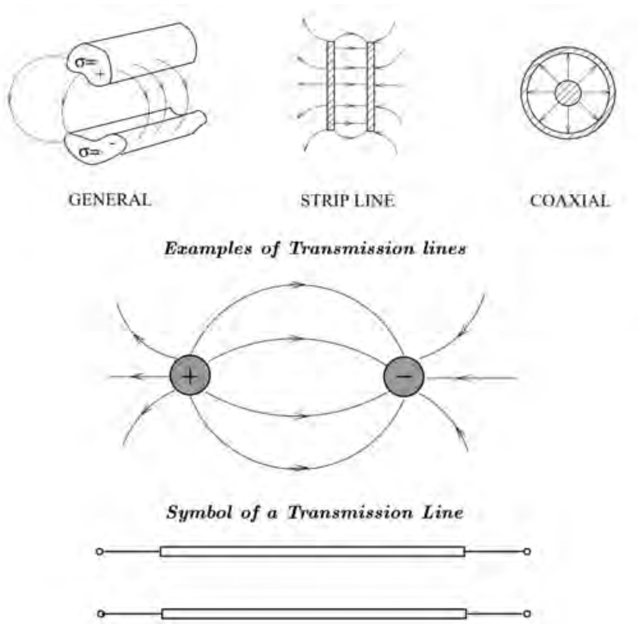

Figure 11.1: Various kinds of transmission lines. Schematically, all of them can be modeled by two parallel wires.

Transmission lines were the first electromagnetic waveguides ever invented. The were driven by the needs in telegraphy technology. It is best to introduce transmission line theory from the viewpoint of circuit theory. This theory is also discussed in many textbooks and lecture notes. Transmission lines are so important in modern day electromagnetic engineering, that most engineering electromagnetics textbooks would be incomplete without introducing the topic [30, 31, 39, 45, 49, 60, 72, 76, 78, 79].

Circuit theory is robust and is not sensitive to the detail shapes of the components involved such as capacitors or inductors. Moreover, many transmission line problems cannot be analyzed with the full form of Maxwell's equations,[1] but approximate solutions can be obtained using circuit theory in the long-wavelength limit. We shall show that circuit theory is an approximation of electromagnetic field theory when the wavelength is very long: the longer the wavelength, the better is the approximation [45].

Examples of transmission lines are shown in Figure 11.1. The symbol for a transmission line is usually represented by two pieces of parallel wires, but in practice, these wires need not be parallel as showin in Figure 11.2.

Figure 11.2: A twisted pair transmission line where the two wires are not parallel to each other (courtesy of slides by A. Wadhwa, A.L. Dal, N. Malhotra [80].)

Circuit theory also explains why waveguides can be made sloppily when wavelength is long or the frequency low. For instance, in the long-wavelength limit, we can make twisted-pair waveguides with abandon, and they still work well (see Figure 11.2). Hence, it is simplest to first explain the propagation of electromagnetic signal on a transmission line using circuit analysis.

11.1.1 Time-Domain Analysis

We will start with performing the time-domain analysis of a simple, infinitely long transmission line. Remember that two pieces of metal can accumulate attractive charges between them, giving rise to capacitive coupling, or electric field, and hence stored energy in the electric field. Moreover, a piece of wire carrying a current generates a magnetic field, and hence, yielding stored energy in the magnetic field. These stored energies are the sources of the capacitive and inductive effects. But these capacitive and inductive effects are distributed over the spatial dimension of the transmission line. Therefore, it is helpful to think of the two pieces of metal as consisting of small segments of metal concatenated together. Each of these segments will have a small inductance, as well as a small capacitive coupling between them. Hence, we can model two pieces of metal with a distributed lumped element model as

[1] Usually called full-wave analysis.

shown in Figure 11.3. For simplicity, we assume the other conductor to be a ground plane, so that it need not be approximated with lumped elements.

In the transmission line, the voltage $V(z,t)$ and $I(z,t)$ are functions of both space z and time t, but we will model the space variation of the voltage and current with discrete step approximation. The voltage varies from node to node while the current varies from branch to branch of the lump-element model.

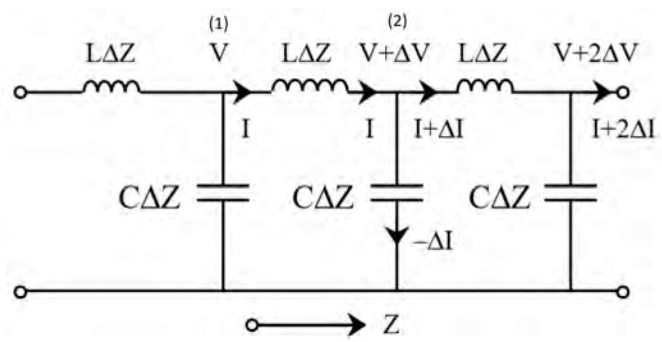

Figure 11.3: A lumped-element approximation of the physics of two parallel pieces of wire. There are capacitive coupling between the wires, as well as that a piece of wire acts like an inductor.

First, we recall that the V-I relation of an inductor is

$$V_0 = L_0 \frac{dI_0}{dt} \tag{11.1.1}$$

where L_0 is the inductor, V_0 is the time-varying voltage drop across the inductor, and I_0 is the current through the inductor. Then using this relation between node 1 and node 2, we have

$$V - (V + \Delta V) = L\Delta z \frac{\partial I}{\partial t} \tag{11.1.2}$$

The left-hand side is the voltage drop across the inductor, while the right-hand side follows from the aforementioned V-I relation of an inductor, but we have replaced $L_0 = L\Delta z$. Here, L is the inductance per unit length (line inductance) of the transmission line. And $L\Delta z$ is the incremental inductance due to the small segment of metal of length Δz. Then the above can be simplified to

$$\Delta V = -L\Delta z \frac{\partial I}{\partial t} \tag{11.1.3}$$

Next, we make use of the V-I relation for a capacitor, which is

$$I_0 = C_0 \frac{dV_0}{dt} \tag{11.1.4}$$

where C_0 is the capacitor, I_0 is the current through the capacitor, and V_0 is a time-varying voltage drop across the capacitor. Thus, applying this relation at node 2 gives

$$-\Delta I = C\Delta z \frac{\partial}{\partial t}(V + \Delta V) \approx C\Delta z \frac{\partial V}{\partial t} \tag{11.1.5}$$

where C is the capacitance per unit length, and $C\Delta z$ is the incremental capacitance between the small piece of metal and the ground plane. In the above, we have used Kirchhoff current law to surmise that the current through the capacitor is $-\Delta I$, where $\Delta I = I(z+\Delta z, t) - I(z, t)$. In the last approximation in (11.1.5), we have dropped a term involving the product of Δz and ΔV, since it will be very small or second order in magnitude.

In the limit when $\Delta z \to 0$, one gets from (11.1.3) and (11.1.5) that

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t} \tag{11.1.6}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t} \tag{11.1.7}$$

The above are the telegrapher's equations.[2] They are two coupled first-order equations, and can be converted into second-order equations easily. Therefore,

$$\frac{\partial^2 V}{\partial z^2} - LC\frac{\partial^2 V}{\partial t^2} = 0 \tag{11.1.8}$$

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = 0 \tag{11.1.9}$$

The above are wave equations that we have previously studied, where the velocity of the wave is given by

$$v = \frac{1}{\sqrt{LC}} \tag{11.1.10}$$

Furthermore, if we assume that

$$V(z,t) = f_+(z - vt) \tag{11.1.11}$$

a right-traveling wave, and substituting it into (11.1.6) yields, after exchanging the right-hand side with the left-hand side that

$$-L\frac{\partial I}{\partial t} = f'_+(z - vt) \tag{11.1.12}$$

where $f'(x) = df(x)/dx$. Substituting $V(z,t)$ into (11.1.7) yields

$$\frac{\partial I}{\partial z} = Cvf'_+(z - vt) \tag{11.1.13}$$

[2]They can be thought of as the distillation of the Faraday's law and Ampere's law from Maxwell's equations without the source term. Their simplicity gives them an important role in engineering electromagnetics.

The above implies that[3]

$$I = \sqrt{\frac{C}{L}} f_+(z - vt) \tag{11.1.14}$$

Consequently,

$$\frac{V(z,t)}{I(z,t)} = \sqrt{\frac{L}{C}} = Z_0 \tag{11.1.15}$$

where Z_0 is the characteristic impedance of the transmission line. The above ratio is only true for one-way traveling wave, in this case, one that propagates in the $+z$ direction.

For a wave that travels in the negative z direction, i.e.,

$$V(z,t) = f_-(z + vt) \tag{11.1.16}$$

with the corresponding $I(z,t)$ derived, one can show that

$$\frac{V(z,t)}{I(z,t)} = -\sqrt{\frac{L}{C}} = -Z_0 \tag{11.1.17}$$

Time-domain analysis is very useful for transient analysis of transmission lines, especially when nonlinear elements are coupled to the transmission line. Another major strength of transmission line model is that it is a simple way to introduce time-delay (also called retardation) in a circuit. Time delay is a wave propagation effect, and it is harder to incorporate into circuit theory or a pure circuit model consisting of R, L, and C.

In circuit theory, Laplace's equation is usually solved, which is equivalent to Helmholtz equation with infinite wave velocity, namely,

$$\lim_{c\to\infty} \nabla^2\Phi(\mathbf{r}) + \frac{\omega^2}{c^2}\Phi(\mathbf{r}) = 0 \quad \Longrightarrow \quad \nabla^2\Phi(\mathbf{r}) = 0 \tag{11.1.18}$$

Hence, events in Laplace's equation happen instantaneously. In other words, circuit theory assumes that the velocity of the wave is infinite, and there is no retardation effect.

11.1.2 Frequency-Domain Analysis

Frequency domain analysis is very popular as it makes the transmission line equations very simple. Moreover, generalization to a lossy system is quite straight forward. Furthermore, for linear time invariant systems, the time-domain signals can be obtained from the frequency-domain data by performing a Fourier inverse transform.

For a time-harmonic signal on a transmission line, one can analyze the problem in the frequency domain using phasor technique. A phasor variable is linearly proportional to a Fourier transform variable. The telegrapher's equations (11.1.6) and (11.1.7) then in frequency domain become

$$\frac{d}{dz}V(z,\omega) = -j\omega L I(z,\omega) \tag{11.1.19}$$

$$\frac{d}{dz}I(z,\omega) = -j\omega C V(z,\omega) \tag{11.1.20}$$

[3] An integration constant independent of t and z can be added, but it can be argued that it is zero.

The corresponding Helmholtz equations are then

$$\frac{d^2V}{dz^2} + \omega^2 LCV = 0 \tag{11.1.21}$$

$$\frac{d^2I}{dz^2} + \omega^2 LCI = 0 \tag{11.1.22}$$

The general solutions to the above are

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z} \tag{11.1.23}$$

$$I(z) = I_+ e^{-j\beta z} + I_- e^{j\beta z} \tag{11.1.24}$$

where $\beta = \omega\sqrt{LC}$. This is similar to what we have seen previously for plane waves in the one-dimensional wave equation in free space, where

$$E_x(z) = E_{0+} e^{-jk_0 z} + E_{0-} e^{jk_0 z} \tag{11.1.25}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. We see much similarity between (11.1.23), (11.1.24), and (11.1.25).

To see the solution in the time domain, we let the phasor $V_\pm = |V_\pm| e^{j\phi_\pm}$, and the voltage signal above can then be converted back to the time domain as

$$V(z,t) = \Re e\{V(z,\omega)e^{j\omega t}\} \tag{11.1.26}$$

$$= |V_+|\cos(\omega t - \beta z + \phi_+) + |V_-|\cos(\omega t + \beta z + \phi_-) \tag{11.1.27}$$

As can be seen, the first term corresponds to a right-traveling wave, while the second term is a left-traveling wave.

Furthermore, if we assume only a one-way traveling wave to the right by letting $V_- = I_- = 0$, then it can be shown that, for a right-traveling wave

$$\frac{V(z)}{I(z)} = \frac{V_+}{I_+} = \sqrt{\frac{L}{C}} = Z_0 \tag{11.1.28}$$

In the above, the telegrapher's equations, (11.1.19) or (11.1.20) have been used to find a relationship between I_+ and V_+.

Similarly, applying the same process for a left-traveling wave only, by letting $V_+ = I_+ = 0$, then

$$\frac{V(z)}{I(z)} = \frac{V_-}{I_-} = -\sqrt{\frac{L}{C}} = -Z_0 \tag{11.1.29}$$

11.2 Lossy Transmission Line

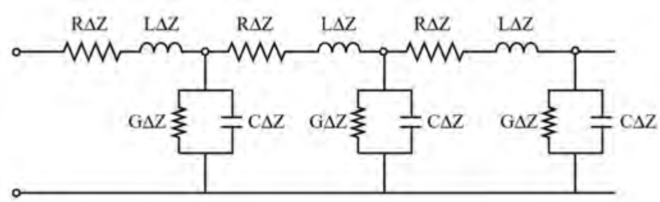

Figure 11.4: In a lossy transmission line, series resistance can be added to the series inductance, and the shunt conductance can be added to the shun susceptance of the capacitor. However, this problem is homomorphic to the lossless case in the frequency domain.

The strength of frequency domain analysis is revealed in the study of lossy transmission lines. The previous analysis, which is valid for lossless transmission line, can be easily generalized to the lossy case in the frequency domain. In using frequency domain and phasor technique, impedances will become complex numbers as shall be shown.

To include loss, we use the lumped-element model as shown in Figure 11.4. One thing to note is that $j\omega L$ is actually the series line impedance of the transmission line, while $j\omega C$ is the shunt line admittance of the line. First, we can rewrite the expressions for the telegrapher's equations in (11.1.19) and (11.1.20) in terms of series line impedance and shunt line admittance to arrive at

$$\frac{d}{dz}V = -ZI \tag{11.2.1}$$

$$\frac{d}{dz}I = -YV \tag{11.2.2}$$

where $Z = j\omega L$ and $Y = j\omega C$. The above can be generalized to the lossy case as shall be shown.

The geometry in Figure 11.4 is homomorphic[4] to the lossless case in Figure 11.3 and in its math structure. Hence, when lossy elements are added in the geometry, we can surmise that the corresponding telegrapher's equations are similar to those above. But to include loss, we generalize the series line impedance and shunt admittance from the lossless case to lossy case as follows:

$$Z = j\omega L \to Z = j\omega L + R \tag{11.2.3}$$

$$Y = j\omega C \to Y = j\omega C + G \tag{11.2.4}$$

[4]A math term for "similar in structure". The term is even used in computer science describing a emerging field of homomorphic computing.

where R is the series line resistance, and G is the shunt line conductance, and now Z and Y are the series impedance and shunt admittance, respectively. Then, the corresponding Helmholtz equations are

$$\frac{d^2V}{dz^2} - ZYV = 0 \tag{11.2.5}$$

$$\frac{d^2I}{dz^2} - ZYI = 0 \tag{11.2.6}$$

or

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0 \tag{11.2.7}$$

$$\frac{d^2I}{dz^2} - \gamma^2 I = 0 \tag{11.2.8}$$

where $\gamma^2 = ZY$, or that one can also think of $\gamma^2 = -\beta^2$ by comparing with (11.1.21) and (11.1.22). Then the above is homomorphic to the lossless case except that now, β is a complex number, indicating that the field is decaying as it propagates. As before, the above are second order one-dimensional Helmholtz equations where the general solutions are

$$V(z) = V_+ e^{-\gamma z} + V_- e^{\gamma z} \tag{11.2.9}$$

$$I(z) = I_+ e^{-\gamma z} + I_- e^{\gamma z} \tag{11.2.10}$$

and

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = j\beta \tag{11.2.11}$$

Hence, $\beta = \beta' - j\beta''$ is now a complex number. In other words,

$$e^{-\gamma z} = e^{-j\beta' z - \beta'' z}$$

is an oscillatory and decaying wave. Or focusing on the voltage case,

$$V(z) = V_+ e^{-\beta'' z - j\beta' z} + V_- e^{\beta'' z + j\beta' z} \tag{11.2.12}$$

Again, letting $V_\pm = |V_\pm| e^{j\phi_\pm}$, the above can be converted back to the time domain as

$$V(z,t) = \Re e\{V(z,\omega)e^{j\omega t}\} \tag{11.2.13}$$

$$= |V_+|e^{-\beta'' z}\cos(\omega t - \beta' z + \phi_+) + |V_-|e^{\beta'' z}\cos(\omega t + \beta' z + \phi_-) \tag{11.2.14}$$

The first term corresponds to a decaying wave moving to the right while the second term is also a decaying wave moving to the left. When there is no loss, or $R = G = 0$, and from (11.2.11), we retrieve the lossless case where $\beta'' = 0$ and $\gamma = j\beta = j\omega\sqrt{LC}$.

Notice that for the lossy case, the characteristic impedance, which is the ratio of the voltage to the current for a one-way wave, can similarly be derived using homomorphism:

$$Z_0 = \frac{V_+}{I_+} = -\frac{V_-}{I_-} = \sqrt{\frac{L}{C}} = \sqrt{\frac{j\omega L}{j\omega C}} \to Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \tag{11.2.15}$$

The above Z_0 is manifestly a complex number. Here, Z_0 is the ratio of the phasors of the one-way traveling waves, and apparently, their current phasor and the voltage phasor will not be in phase for lossy transmission line.

In the absence of loss, the above again becomes

$$Z_0 = \sqrt{\frac{L}{C}} \tag{11.2.16}$$

the characteristic impedance for the lossless case previously derived.